Solution 1.

The Lagrangian

The Lagrangian for this problem is:

$$\mathcal{L}(x, \lambda, \mu) = \sum_{i=1}^{n} x_i^2 + \lambda \left(c - \sum_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} \mu_i x_i$$

where:

- λ is the multiplier for the equality constraint $\sum_{i=1}^{n} x_i = c$,
- $\mu_i \geq 0$ are the multipliers for the inequality constraints $x_i \geq 0$.

KKT Conditions

The KKT conditions are:

1. **Stationarity:**
$$\frac{\partial \mathcal{L}}{\partial x_i} = 2x_i - \lambda - \mu_i = 0, \quad \forall i.$$
2. **Primal feasibility:**
$$\sum_{i=1}^{n} x_i = c, \quad x_i \geq 0, \quad \forall i.$$
3. **Dual feasibility:**
$$\mu_i \geq 0, \quad \forall i.$$
4. **Complementary slackness:**
$$\mu_i x_i = 0, \quad \forall i.$$

Solving the KKT Conditions

- From the stationarity conditions:
$$\mu_i = 2x_i - \lambda$$
- From complementary slackeness conditions
$$\mu_i x_i = 0, \forall i, \implies (2x_i - \lambda)x_i$$
  This implies that for each i,
  either $x_i = 0$
  or $\mu_i = 0$ $(x_i > 0)$, then: $x_i = \frac{\lambda}{2}$.

- From the equality constraint:

$$\sum_{i=1}^{n} x_i = c.$$

If all $x_i > 0$, then $x_i = \frac{\lambda}{2}, \forall i$

substitute $x_i = \frac{\lambda}{2}$ into $\sum_{i=1}^{n} x_i = c$:

$$n \cdot \frac{\lambda}{2} = c \implies \lambda = \frac{2c}{n}.$$

- Substitute λ back into x_i:

$$x_i = \frac{\lambda}{2} = \frac{c}{n}, \quad \forall i.$$

- feasibility:
  - $x_i = \frac{c}{n} \geq 0$ for all i since $c > 0$,
  - $\sum_{i=1}^{n} x_i = n \cdot \frac{c}{n} = c$.

If some x's are zero, let k be the number of variables $x_i > 0$ then, we have from the equality constraint, by substituting λ back into x_i:

$$\sum_{i=1}^{n} x_i = c \implies k \cdot \frac{\lambda}{2} = c \implies \lambda = \frac{2c}{k}.$$

So for the k positive variables: $x_i = \frac{\lambda}{2} = \frac{c}{k}, \quad$ for $i = 1, \ldots, k$. and for the remaining $n - k$ variables: $x_i = 0$.

Solution 2. Minimize:

$$f(x) = x^\top Q x + c^\top x$$

subject to:

$$Ax = b, \quad x \geq \mathbf{0},$$

where:

- $x \in \mathbb{R}^n$ are the decision variables,
- $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix (ensuring convexity of $f(x)$),
- $c \in \mathbb{R}^n$ is a coefficient vector,
- $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ represent linear equality constraints,
- $x \geq \mathbf{0}$ ensures non-negativity of each component of x.

Lagrangian

The Lagrangian is:

$$\mathcal{L}(x, \lambda, \mu) = x^\top Q x + c^\top x + \lambda^\top (b - Ax) - \mu^\top x,$$

where:

- $\lambda \in \mathbb{R}^m$ is the vector of Lagrange multipliers for the equality constraints,
- $\mu \geq \mathbf{0} \in \mathbb{R}^n$ is the Lagrange multiplier vector for the inequality constraint $x \geq 0$.

KKT Conditions

The KKT conditions are:

1. **Stationarity:**

$$\nabla_x \mathcal{L} = 2Qx + c - A^\top \lambda - \mu = \mathbf{0}$$

   which gives:

$$2Qx + c = A^\top \lambda + \mu.$$

2. **Primal feasibility:**

$$Ax = b, \quad x \geq \mathbf{0}.$$

3. **Dual feasibility:**

$$\mu \geq \mathbf{0}.$$

4. **Complementary slackness:**

$$\mu^\top x = 0.$$

Example

Minimize:

$$f(x) = x^\top \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} -4 \\ -6 \end{bmatrix}^\top x,$$

Subject to:

$$\begin{bmatrix} 1 & 1 \end{bmatrix} x = 4, \quad x \geq \mathbf{0}.$$

The **Lagrangian** is:

$$\mathcal{L}(x, \lambda, \mu) = x^\top Q x + c^\top x + \lambda(b - Ax) - \mu^\top x,$$

where:

- $\lambda \in \mathbb{R}$ is the multiplier for the equality constraint $Ax = b$,
- $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \geq \mathbf{0}$ are the multipliers for the inequality constraint $x \geq \mathbf{0}$.

KKT Conditions

The Karush-Kuhn-Tucker (KKT) conditions are:

1. **Stationarity**:

$$\nabla_x \mathcal{L}(x, \lambda, \mu) = 2Qx + c - A^\top \lambda - \mu = 0.$$

   Substituting Q, c, and A:

$$2\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} -4 \\ -6 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \lambda - \mu = 0,$$

   which simplifies to:

$$\begin{bmatrix} 4x_1 - 4 \\ 4x_2 - 6 \end{bmatrix} = \begin{bmatrix} \lambda + \mu_1 \\ \lambda + \mu_2 \end{bmatrix}.$$

2. **Primal Feasibility**:

$$x_1 + x_2 = 4, \quad x \geq \mathbf{0}.$$

3. **Dual Feasibility**:

$$\mu_1 \geq 0, \quad \mu_2 \geq 0.$$

4. **Complementary Slackness**:

$$\mu_1 x_1 = 0, \quad \mu_2 x_2 = 0.$$

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