A regression model with a hidden logistic process for feature extraction from time series

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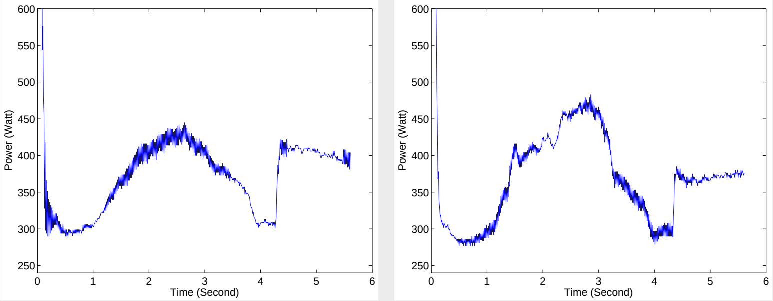
Context: Feature extraction from the switch operation signals

- Signals of the consumed electrical power during switch operations



- Each switch operation consists of 5 successive electromechanical motions
- The signals present smooth or abrupt changes between different regimes
- The proposed solution: use an adapted regression model whose parameters will be used as the feature vector for each signal.

Piecewise polynomial regression model [McGee & Carleton 70]

- The data: $\{(x_1, t_1), \ldots, (x_n, t_n)\}$
  - x_i: real dependant variable: the observation of the signal
  - t_i: independant variable representing the time
- The piecewise polynomial regression model generating the signal $\mathbf{x}$ is:

$$\forall i = 1, \ldots, n, \quad x_i = \boldsymbol{\beta}_k^T \boldsymbol{r}_i + \sigma_k \epsilon_i \quad ; \quad \epsilon_i \sim \mathcal{N}(0, 1)$$

  - k satisfies $i \in I_k = (\gamma_k, \gamma_{k+1}]$: indexes of elements in segment k
  - $\boldsymbol{r}_i = (1, t_i, \ldots, t_i^p)^T$: time-dependant covariate vector in R^{p+1}
  - $\boldsymbol{\beta}_k$: regression coefficients vector $\in R^{(p+1)}$ for the k^{th} segment

The model parameters

$(\boldsymbol{\psi}, \boldsymbol{\gamma})$ with $\boldsymbol{\psi} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K, \sigma_1^2, \ldots, \sigma_K^2)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{K+1})$.

Parameter estimation for the piecewise regression model

Maximize the likelihood of (ψ, γ) or equivalently minimize, with respect to (ψ, γ):

$$J(\psi, \gamma) = \sum_{k=1}^{K} \sum_{i \in I_k} \left[\log \sigma_k^2 + \frac{(x_i - \boldsymbol{\beta}_k^T \boldsymbol{r}_i)^2}{\sigma_k^2} \right].$$

- Global optimization using Fisher's algorithm [Fisher 58] based on dynamic programming [Bellman 61; Lechevallier 90] since the criterion J is additive on k
- Local optimization using an iterative variant of Fisher's Algorithm [Samé et al. 07]

Time series approximation and segmentation

- $\hat{x}_i = \sum_{k=1}^{K} \hat{z}_{ik} \hat{\boldsymbol{\beta}}_k^T \boldsymbol{r}_i \quad ; \quad \forall i = 1, \ldots, n$
- $\hat{z}_{ik} = 1$ if $i \in (\hat{\gamma}_k, \hat{\gamma}_{k+1}]$ (x_i belongs to the k^{th} segment) and $\hat{z}_{ik} = 0$ otherwise

- Using dynamic programming can be computationally expensive
- Provides a hard partition ⇒ adapted for regimes with abrupt changes

The proposed regression based on hidden process approach

The global regression model

$$\forall i = 1, \ldots, n, , \quad x_i = \boldsymbol{\beta}_{z_i}^T \boldsymbol{r}_i + \sigma_{z_i} \epsilon_i \quad ; \quad \epsilon_i \sim \mathcal{N}(0, 1),$$

- $z_i \in \{1, \ldots, K\}$ hidden variable: the class label of the regression model generating x_i

$\mathbf{z} = (z_1, \ldots, z_n)$ is a hidden discrete process

$z_i \sim \mathcal{M}\left(1, \pi_{i1}(\mathbf{w}), \ldots, \pi_{iK}(\mathbf{w})\right)$; where

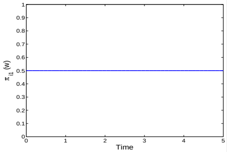
$$\pi_{ik}(\mathbf{w}) = p(z_i = k; \mathbf{w}) = \frac{\exp\left(\boldsymbol{w}_k^T \boldsymbol{v}_i\right)}{\sum_{\ell=1}^{K} \exp\left(\boldsymbol{w}_\ell^T \boldsymbol{v}_i\right)},$$

- $\boldsymbol{v}_i = (1, t_i, \ldots, t_i^q)^T$ time-dependant covariate vector $\in R^{(q+1)}$
- $\mathbf{w} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_K)$ the parameter vector for the K k logistic functions $\in R^{K \times (q+1)}$

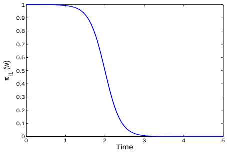
Flexibility of the logistic transformation: Example for $K = 2$:

1. $\pi_{ik}(\mathbf{w})$ in relation to the dimension q of $\boldsymbol{w}_k$:

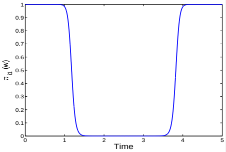
$q = 0$



$q = 1$



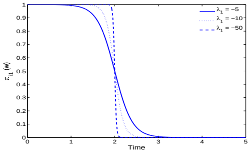
$q = 2$



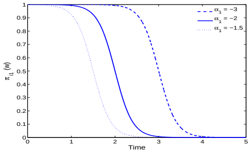
$\Rightarrow q = 1$ guarantees segmentation into contiguous segments

2. $\pi_{ik}(\mathbf{w})$ in relation to $\boldsymbol{w}_k$ for $q = 1$; we parametrize $\boldsymbol{w}_k$ by $\boldsymbol{w}_k = \lambda_k(\alpha_k, 1)^T$

$\alpha_1 = -2$



$\lambda_1 = -5$



$\Rightarrow$ The parameter λ_k controls the quality of transitions between classes
$\Rightarrow$ The parameter α_k controls the transition time point.

Parameter estimation by maximum likelihood

- Derived mixture density

$$p(x_i;\boldsymbol{\theta}) = \sum_{k=1}^{K} \pi_{ik}(\mathbf{w})\mathcal{N}(x_i;\boldsymbol{\beta}_k^T \boldsymbol{r}_i, \sigma_k^2)$$

- Model parameters

$$\boldsymbol{\theta} = \big(\mathbf{w}, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_K, \sigma_1^2, \ldots, \sigma_K^2\big)$$

- Log-likelihood of $\boldsymbol{\theta}$:

$$L(\boldsymbol{\theta};\mathbf{x}) = \sum_{i=1}^{n} \log \sum_{k=1}^{K} \pi_{ik}(\boldsymbol{w})\mathcal{N}(x_i;\boldsymbol{\beta}_k^T \boldsymbol{r}_i, \sigma_k^2).$$

- Maximization of $L(\boldsymbol{\theta};\mathbf{x})$ by a dedicated Expectation-Maximization (EM) algorithm [Dempster et al. 77].

Dedicated EM algorithm

Initialization: $\boldsymbol{\theta}^{(0)}$

Repeat until convergence:

1. **E step: Expectation** (at iteration m)

   Compute the conditional expectation of the complete log-likelihood $L(\boldsymbol{\theta};\mathbf{x},\mathbf{z})$

$$Q(\boldsymbol{\theta},\boldsymbol{\theta}^{(m)}) = E\Big[L(\boldsymbol{\theta};\mathbf{x},\mathbf{z})\Big|\mathbf{x},\boldsymbol{\theta}^{(m)}\Big]$$

$$= \underbrace{\sum_{i=1}^{n}\sum_{k=1}^{K}\tau_{ik}^{(m)}\log \pi_{ik}(\mathbf{w})}_{Q_1(\mathbf{w})} + \underbrace{\sum_{i=1}^{n}\sum_{k=1}^{K}\tau_{ik}^{(m)}\log \mathcal{N}\Big(x_i;\boldsymbol{\beta}_k^T\boldsymbol{r}_i,\sigma_k^2\Big)}_{Q_2(\boldsymbol{\beta}_k,\sigma_k^2;k=1,\ldots,K)},$$

2. **M step: Maximization** (at iteration m)

   Compute $\boldsymbol{\theta}^{(m+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta},\boldsymbol{\theta}^{(m)})$

Details of the M step

1. Maximization of Q_2 with respect to $\boldsymbol{\beta}_k$s: Analytic solutions of K separate polynomial regression problems weighted by the $\tau_{ik}^{(m)}$s:
   - $\beta_k^{(m+1)} = (\mathbf{M}^T \boldsymbol{\Gamma}_k^{(m)} \mathbf{M})^{-1} \mathbf{M}^T \boldsymbol{\Gamma}_k^{(m)} \mathbf{x}$

   where $\mathbf{M}$ is the design matrix and $\boldsymbol{\Gamma}_k^{(m)} = \text{diag}\,(\tau_{1k}^{(m)}, \ldots, \tau_{nk}^{(m)})$.

   Maximization of Q_2 with respect to σ_k^2s
   - $\sigma_k^{2(m+1)} = \frac{1}{\sum_{i=1}^n \tau_{ik}^{(m)}} \sum_{i=1}^n \tau_{ik}^{(m)} (x_i - \boldsymbol{\beta}_k^{T(m+1)} \boldsymbol{r}_i)^2.$

2. Maximize Q_1 with respect to $\mathbf{w}$: Solve a multiclass convex logistic regression problem weighted by the $\tau_{ik}^{(m)}$s $\Rightarrow$ IRLS algorithm [Chen 99, Green 84, Krishnapuram 05]

$$\mathbf{w}^{(c+1)} = \mathbf{w}^{(c)} - \left[\frac{\partial^2 Q_1(\mathbf{w})}{\partial \mathbf{w} \partial \mathbf{w}^T}\right]^{-1}_{\mathbf{w}=\mathbf{w}^{(c)}} \frac{\partial Q_1(\mathbf{w})}{\partial \mathbf{w}}\Big|_{\mathbf{w}=\mathbf{w}^{(c)}} ,$$

$\Rightarrow$ Applying the IRLS algorithm provides the parameter $\mathbf{w}^{(m+1)}$.

Time series approximation and segmentation with the proposed model

Time series approximation

- As in standard regression, given the estimated parameters, x_i is approximated by its expectation $\forall i = 1, \ldots, n$:

$$\hat{x}_i \quad = \quad E(x_i; \hat{\boldsymbol{\theta}}) = \int_R x_i p(x_i; \hat{\boldsymbol{\theta}}) dx_i = \sum_{k=1}^{K} \pi_{ik}(\hat{\mathbf{w}}) \hat{\boldsymbol{\beta}}_k^T \boldsymbol{r}_i \quad .$$

A sum of polynomials weighted by the logistic probabilities $\pi_{ik}(\hat{\mathbf{w}})$'s.

$\Rightarrow$ Adapted for a smooth or abrupt transitions between the regression models.

Time series segmentation

- The estimated class label $\hat{z}_i$ of x_i is given by the rule:

$$\hat{z}_i = \arg \max_{1 \leq k \leq K} \pi_{ik}(\hat{\mathbf{w}}).$$

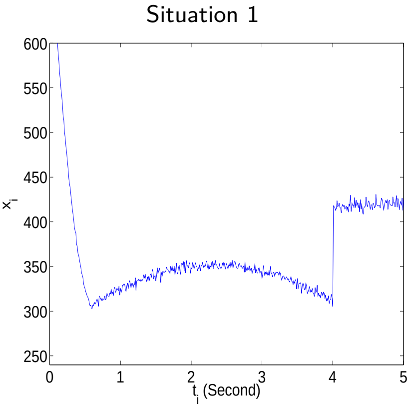
Experiments using simulated data

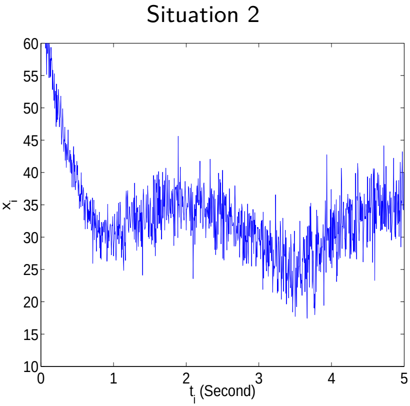
- Evaluation criteria:
  1. Misclassification error rate (segmentation error)
  2. Error between the true simulated curve without noise and the estimated curve (Denoising error):

$$\frac{1}{n}\sum_{i=1}^{n}(E(x_i;\boldsymbol{\theta}) - E(x_i;\hat{\boldsymbol{\theta}}))^2$$

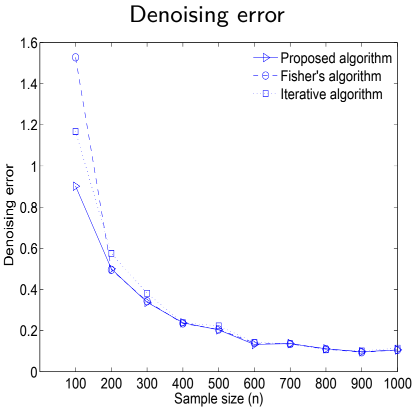
- Comparison with the two piecewise regression approahces
- 2 situations of signals with
  - $K = 3$, $p = 2$, $q = 1$
  - Varying the sample size $n = 100, 200, \ldots, 1000$
  - $\sigma_1^2 = 4$, $\sigma_2^2 = 10$, $\sigma_3^2 = 15$
- Assessment criteria are averaged over 20 samples for each value of n

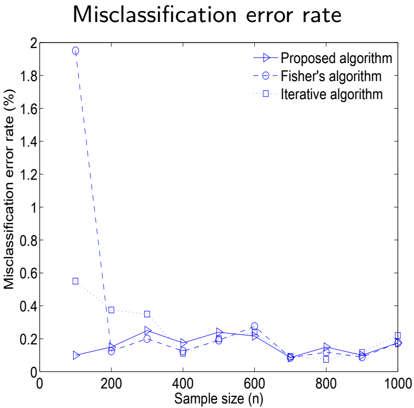
Example of simulated signals



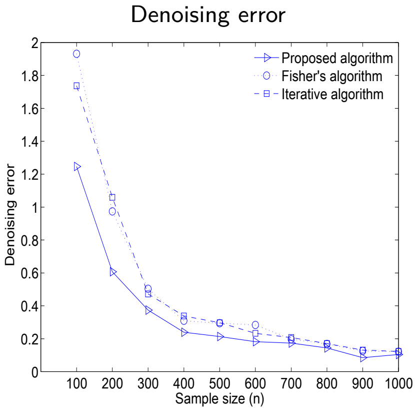


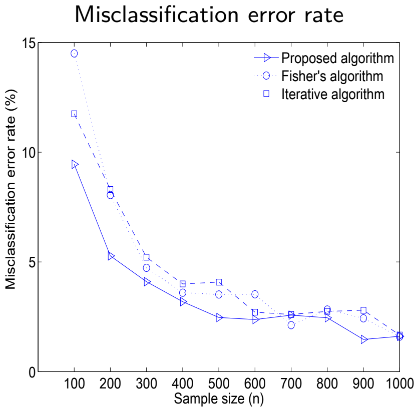
Results 1 (Situation 1)



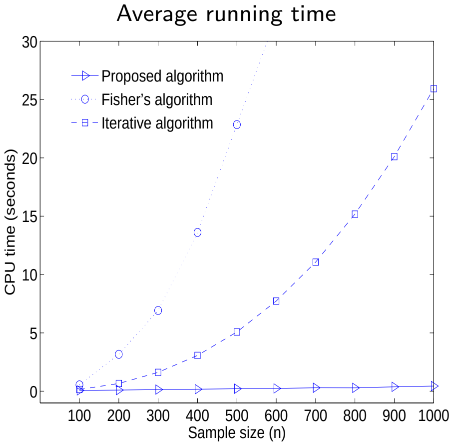


Results 2 (Situation 2)





Computing time

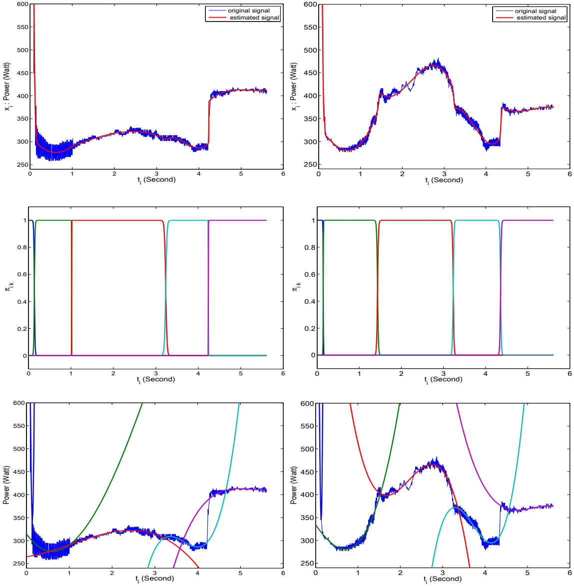


Application to real signals

Original signal and estimated signal

Probabilities of the regression models

Corresponding regression models



Conclusion

- In contrast with the basic polynomial regression, the proposed approach authorizes the regression parameters to vary over time

  $\Rightarrow$ Accurate modeling of nonlinear signals

- The proposed model integrates a logistic process which makes possible to change smoothly within various possible regression models

- In addition to feature extraction, this approach can be used to denoise and segment time series (or signals)

- Computationally efficient.

Thank you!